Quantum correlations VIII

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- PPT states:

- In Quantum Computing, a characterization of states with positive partial transposition (PPT states) is of paramount importance.

- Namely, in the fifth lecture, we have seen

$$\mathfrak{S}_{sep} \subset \mathfrak{S}_{PPT} \subset \mathfrak{S}, \tag{1}$$

and the family of PPT (transposable) states on $B(\mathcal{H}) \otimes B(\mathcal{K})$ is defined as

$$\mathfrak{S}_{PPT} = \{\varphi \in \mathfrak{S} : \varphi \circ (id_{B(\mathcal{H})} \otimes \tau_{\mathcal{K}}) \in \mathfrak{S}\} \tag{2}$$

where, as before, $\tau_{\mathcal{K}}$ stands for the transposition map, now defined on $B(\mathcal{K})$.

- Thus, states in $\mathfrak{S}_{PPT}$ contain non-classical correlations.
- However, it is believed that genuine quantum correlations are contained in states from $\mathfrak{S} \setminus \mathfrak{S}_{PPT}$.
- This makes clear our motivation in description of PPT states.
- Observe that to define this family of states, we have used properties of positive maps.
- On the other hand, in early days of attempts to classify the structure of positive maps, Choi observed, in 1982, that the transposition is playing a distinguished role in describing tensor product of matrices which are positive.
- We remind that the theory of positive maps is not complete, although hard work done in last decades.

- However, Choi noted that the theory (*of positive maps*) is easy for dimension 2 but very non-trivial for higher dimensions. In particular, he observed the transposition is playing an important role.

- The significance of transposition was considered in Physics by Peres and Horodecki's, in 1996, and their research led to the first classification of quantum states in Quantum Information.

- Then, further investigations have shown the importance of PPT states for quantum computing.

- Here, we wish to present a characterization of PPT states for quantum systems. To this end we will exploit links between tensor products and properly chosen mapping spaces.

- This should be expected as for the definition of PPT states the theory of positive mappings was used.

- On the other hand, this theory is related to tensor products, and finally, the Grothendieck approach to tensor products is emerging from rules of quantization (Born rule!), as it was described in the fourth lecture.

- In particular, we have seen:

$$L(\mathfrak{A}, \mathfrak{A}) \cong (\mathfrak{A} \otimes_\pi \mathfrak{A}_*)^*,$$

where $L(\mathfrak{A}, \mathfrak{A})$ stands for the set of all bounded linear maps from $\mathfrak{A}$ to $\mathfrak{A}$, and

$$\mathfrak{B}(\mathfrak{A}, \mathfrak{A}_*) \cong L(\mathfrak{A}, \mathfrak{A})$$

where $\mathfrak{B}(\mathfrak{A}, \mathfrak{A}_*)$ denotes the space of bounded bilinear forms on $\mathfrak{A} \times \mathfrak{A}_*$.

Here, $\mathfrak{A}$ stands for the specific algebra generated by observables while $\mathfrak{A}_*$ denotes its predual.

- Let us specify the above results for $\mathfrak{A} \equiv B(\mathcal{H})$. Obviously, then $\mathfrak{A}_*$ is the set of all trace class operators on the Hilbert space $\mathcal{H}$.

- Thus, we will be concerned with Dirac's formalism of Quantum Mechanics.

- We begin with the following result:

- *This result is originated from the Grothendieck theory of tensor products of Banach spaces. However the form given below is taken from Størmer.*

- **Proposition 1.** *There is an isometric isomorphism* $\varphi \to \hat{\varphi}$ *between* $L(B(\mathcal{K}), B(\mathcal{H}))$ *and* $(B(\mathcal{K}) \otimes_\pi \mathcal{F}_T(\mathcal{H}))^*$ *such that*

$$\hat{\varphi}(\sum_{i=1}^{N} A_i \otimes B_i) = \sum_{i=1}^{N} Tr(\varphi(A_i) B_i^t) \tag{3}$$

where $B_i^t = \tau(B_i)$ *(*τ *stands for the transposition),* $\mathcal{F}_T(\mathcal{H})$ *denotes the trace class operators on* $\mathcal{H}$*, and* $\sum_{i=1}^{N} A_i \otimes B_i \in B(\mathcal{K}) \otimes \mathcal{F}_T(\mathcal{H})$*.*

Furthermore $\varphi \in L(B(\mathcal{K}), B(\mathcal{H}))^+$ *if and only if* $\hat{\varphi}$ *is positive on* $B(\mathcal{K})^+ \otimes_\pi \mathcal{F}_T(\mathcal{H})^+$*, where* $L(B(\mathcal{K}), B(\mathcal{H}))^+$ *denotes those linear bounded maps which are positive, i.e* $\varphi(A^* A) \geq 0$ *for* $A \in B(\mathcal{K})$*.*

- We emphasize that, here and in the reminder of this section, both Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are, in general, *infinite dimensional.*

- As PPT states are “dual” to decomposable maps we need an adaptation of the above Proposition for CP and co-CP maps.

- It can be shown, using the identification of $B(\mathcal{H})$ with $\mathcal{F}_T(\mathcal{H})$, the the above modification can be carried out:

- **Theorem 2.** *(1) Let $L(B(\mathcal{H}), B(\mathcal{K})_*)$ stands for the set of all linear bounded, normal (∗-weak continuous) maps from $B(\mathcal{H})$ into $B(\mathcal{K})_*$. There is an isomorphism $\psi \longmapsto \hat{\psi}$ between $L(B(\mathcal{H}), B(\mathcal{K})_*)$ and $(B(\mathcal{H}) \otimes B(\mathcal{K}))_*$ given by*

$$\hat{\psi}(\sum_{i=1}^{N} A_i \otimes B_i) = \sum_{i=1}^{N} Tr(\psi(A_i)B_i^t) \tag{4}$$

The isomorphism is isometric if $\hat{\psi}$ is considered on $B(\mathcal{H}) \otimes_\pi B(\mathcal{K})$.

Furthermore, $\hat{\psi}$ is positive on $(B(\mathcal{H}) \otimes B(\mathcal{K}))^+$ if and only if ψ is completely positive.

(2) There is an isomorphism $\psi \longmapsto \hat{\psi}$ *between* $L\left(B\left(\mathcal{H}\right), B(\mathcal{K})_*\right)$ *and* $\left(B(\mathcal{H}) \otimes B(\mathcal{K})\right)_*$ *given by*

$$\hat{\psi}\left(\sum_i A_i \otimes B_i\right) = \sum_i Tr_{\mathcal{K}} \psi\left(A_i\right) B_i, \quad a_i \in B\left(\mathcal{H}\right), \ b_i \in B\left(\mathcal{K}\right). \quad (5)$$

This isomorphism is isometric if $\hat{\psi}$ *is considered on* $B(H) \otimes_\pi B(K)$*.*

Furthermore $\hat{\psi}$ *is positive on* $(B(\mathcal{H}) \otimes B(\mathcal{K}))^+$ *if and only if* ψ *is complete co-positive.*

- To clarify the theorem we make:

Remark 3. *Since the projective norm* $\| \cdot \|_\pi$ *is submultiplicative, the involution* * *is isometric for this norm it follows that* $B(H) \otimes_\pi B(K)$ *is a* **-Banach algebra.*

Consequently, the concept of positivity is well defined in $B(H) \otimes_\pi B(K)$.

Therefore, in above Theorem, one can combine isometricity with positivity of functionals.

- This result is the "starting point" in the study of so called entanglement mappings.
- Following Belavkin-Ohya scheme, let us take an arbitrary normal state ω on $B(\mathcal{H}) \otimes B(\mathcal{K})$.
- We fix a density matrix ρ_ω describing the composite state ω:

$$\omega(\cdot) \equiv \mathrm{Tr}(\rho_\omega \cdot).$$

- Having the density matrix ρ_ω we will exploit its spectral decomposition.
- Namely:

- Let the spectral decomposition of ρ_ω be

$$\rho_\omega = \sum_{i=1}^{N} \lambda_i \left| e_i \right\rangle \left\langle e_i \right| \tag{6}$$

where $\{e_i\}$ is an orthogonal system in $\mathcal{H} \otimes \mathcal{K}$.

- Define a map $T_\xi : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$

by

$$T_\xi \eta = \xi \otimes \eta \tag{7}$$

- So it is an *embedding (inclusion map)*.

- Then, following *Belavkin-Ohya scheme*, we define the entanglement operator $H : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{K}$

  by
$$H\eta = \sum_i \lambda_i^{\frac{1}{2}} \left(J_{\mathcal{H}\otimes\mathcal{K}} \otimes T^*_{J_{\mathcal{H}}\eta} \right) e_i \otimes e_i \tag{8}$$
  where $e_i \in \mathcal{H} \otimes \mathcal{K}$ for each i, and $J_{\mathcal{H}\otimes\mathcal{K}}$ is a complex conjugation defined by
$$J_{\mathcal{H}\otimes\mathcal{K}} \left(\sum_i (\dot{e}_i, f)\dot{e}_i \right) = \sum_i \overline{(\dot{e}_i, f)}\dot{e}_i \tag{9}$$
  where $\{\dot{e}_i\}$ is the extension (if necessary) of the orthonormal system $\{e_i\}$ to the complete orthogonal normal system $\{\dot{e}_i\}$ in $\mathcal{H} \otimes \mathcal{K}$. $\{e_i\}$ *are taken from the spectral decomposition of* ρ_ω.

- $J_{\mathcal{H}}$ is defined analogously using the spectral resolution of H^*H.

- The definition of the entanglement operator H is a necessary step to define the entanglement mapping:

  $\phi : B(\mathcal{K}) \to B(\mathcal{H})_*$

  given by

$$\phi(B) = \left(H^*(\mathbb{1} \otimes B)H\right)^t = J_{\mathcal{H}} H^*(\mathbb{1} \otimes B)^* H J_{\mathcal{H}}. \tag{10}$$

  where $B \in B(\mathcal{K})$.

- Properties of the entanglement mapping

  ϕ as well as its dual ϕ^* are contained in the next proposition.

Proposition 4. *1. The dual of the entanglement mapping* $\phi^* : B(\mathcal{H}) \to B(\mathcal{K})_*$ *has the following form*

$$\phi^*(A) = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{K}} H A^t H^*, \quad A \in B(\mathcal{H}).$$

2. A state ω *on* $B(\mathcal{H} \otimes \mathcal{K})$ *can be written as*

$$\omega(A \otimes B) = \mathrm{Tr}_{\mathcal{H}} A\phi(B) = \mathrm{Tr}_{\mathcal{K}} B\phi^*(A). \tag{11}$$

- We recall, the definition of PPT states is saying that any such state composed with the partial transposition is again a state.
- This fact combined with the above proposition leads to:

- **Theorem 5.** *PPT states are completely characterized by the mapping ϕ^* which is both CP and co-CP.*

- Taking an arbitrary fixed normal state ω, one has its density matrix ρ_ω.

- This density matrix gives rise to the entanglement mapping ϕ.

- Then, Theorem 5 says that positivity properties of the entanglement mapping ϕ encodes PPT characterization of the given state ω.

- CONCLUSIONS:

- In the given lectures, we presented a concise scheme for quantization and analysis of one of the fundamental concepts of probability calculus – the idea of correlations.

- Although, the quantization of coefficient of correlation is straightforward, the non-commutative setting offers new phenomenon.

- We have observed that a state (which can be understood as a non-commutative integral) does not possess the weak*- Riemann approximation property.

- This implies new type correlations, which are called quantum correlations.

- To study this new type correlations we have used the decomposition theory as well as the selected results from the tensor product theory of metric spaces.

- The utility of decomposition theory stems from the well known fact that the set of states in quantum mechanics does not form a simplex.

- Deep Grothendieck's results were necessary to provide the definition of separable (entangled) density matrices and to study the important class of states – PPT states.

- *In particular, note that the proper geometry for the collection of density matrices, for which people were looking for many years, is that given by the projective norm.*